5.2 ■ THE BIOT-SAVART LAW

5.2.1 ■ Steady Currents

Stationary charges produce electric fields that are constant in time; hence the term **electrostatics**.[8] *Steady currents* produce magnetic fields that are constant in time; the theory of steady currents is called **magnetostatics**.

Stationary charges	$\Rightarrow$	**constant electric fields: electrostatics.**
Steady currents	$\Rightarrow$	**constant magnetic fields: magnetostatics.**

By **steady current** I mean a continuous flow that has been going on forever, without change and without charge piling up anywhere. (Some people call them "stationary currents"; to my ear, that's a contradiction in terms.) Formally, electro/magnetostatics is the régime

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{0}, \tag{5.32}$$

at all places and all times. Of course, there's no such thing in practice as a *truly* steady current, any more than there is a *truly* stationary charge. In this sense, both electrostatics and magnetostatics describe artificial worlds that exist only in textbooks. However, they represent suitable *approximations* as long as the actual fluctuations are remote, or gradual—in fact, for most purposes magnetostatics applies very well to household currents, which alternate 120 times a second!

[8]Actually, it is not necessary that the charges be stationary, but only that the charge *density* at each point be constant. For example, the sphere in Prob. 5.6(b) produces an electrostatic field $1/4\pi\epsilon_0(Q/r^2)\hat{\mathbf{r}}$, even though it is rotating, because ρ does not depend on t.

Notice that a moving *point* charge *cannot possibly constitute a steady current*. If it's here one instant, it's gone the next. This may seem like a minor thing to *you*, but it's a major headache for *me*. I developed each topic in electrostatics by starting out with the simple case of a point charge at rest; then I generalized to an arbitrary charge distribution by invoking the superposition principle. This approach is not open to us in magnetostatics because a moving point charge does not produce a static field in the first place. We are *forced* to deal with extended current distributions right from the start, and, as a result, the arguments are bound to be more cumbersome.

When a steady current flows in a wire, its magnitude I must be the same all along the line; otherwise, charge would be piling up somewhere, and it wouldn't be a steady current. More generally, since $\partial\rho/\partial t = 0$ in magnetostatics, the continuity equation (5.29) becomes

$$\nabla \cdot \mathbf{J} = 0. \tag{5.33}$$

5.2.2 ■ The Magnetic Field of a Steady Current

The magnetic field of a steady line current is given by the **Biot-Savart law**:

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.} \tag{5.34}$$

The integration is along the current path, in the direction of the flow; $d\mathbf{l}'$ is an element of length along the wire, and $\boldsymbol{\mathscr{r}}$, as always, is the vector from the source to the point $\mathbf{r}$ (Fig. 5.17). The constant μ_0 is called the **permeability of free space**:[9]

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}. \tag{5.35}$$

These units are such that $\mathbf{B}$ itself comes out in newtons per ampere-meter (as required by the Lorentz force law), or **teslas** (T):[10]

$$1\ \mathrm{T} = 1\ \mathrm{N/(A \cdot m)}. \tag{5.36}$$

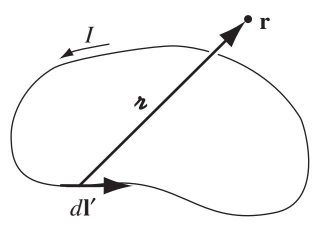


FIGURE 5.17

[9]This is an exact number, not an empirical constant. It serves (via Eq. 5.40) to define the ampere, and the ampere in turn defines the coulomb.

[10]For some reason, in this one case the cgs unit (the **gauss**) is more commonly used than the SI unit: 1 tesla = 10^4 gauss. The earth's magnetic field is about half a gauss; a fairly strong laboratory magnetic field is, say, 10,000 gauss.

As the starting point for magnetostatics, the Biot-Savart law plays a role analogous to Coulomb's law in electrostatics. Indeed, the $1/\mathscr{r}^2$ dependence is common to both laws.

Example 5.5. Find the magnetic field a distance s from a long straight wire carrying a steady current I (Fig. 5.18).

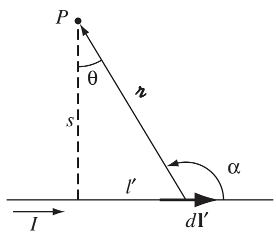


FIGURE 5.18

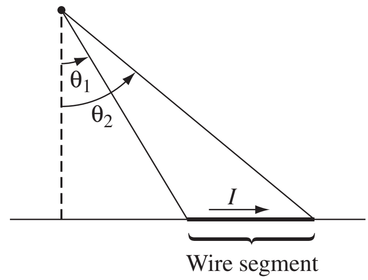


FIGURE 5.19

Solution

In the diagram, $(d\mathbf{l}' \times \hat{\mathscr{r}})$ points *out* of the page, and has the magnitude

$$dl' \sin\alpha = dl' \cos\theta.$$

Also, $l' = s\tan\theta$, so

$$dl' = \frac{s}{\cos^2\theta}\, d\theta,$$

and $s = \mathscr{r}\cos\theta$, so

$$\frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}.$$

Thus

$$\begin{aligned} B &= \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\, d\theta \\ &= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1). \end{aligned} \tag{5.37}$$

Equation 5.37 gives the field of any straight segment of wire, in terms of the initial and final angles θ_1 and θ_2 (Fig. 5.19). Of course, a finite segment by itself could never support a steady current (where would the charge go when it got to the end?), but it might be a *piece* of some closed circuit, and Eq. 5.37 would then represent its contribution to the total field. In the case of an *infinite* wire, $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$, so we obtain

$$B = \frac{\mu_0 I}{2\pi s}. \tag{5.38}$$

Notice that the field is inversely proportional to the distance from the wire—just like the electric field of an infinite line charge. In the region *below* the wire, **B** points *into* the page, and in general, it "circles around" the wire, in accordance with the right-hand rule (Fig. 5.3):

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}. \tag{5.39}$$

As an application, let's find the force of attraction between two long, parallel wires a distance d apart, carrying currents I_1 and I_2 (Fig. 5.20). The field at (2) due to (1) is

$$B = \frac{\mu_0 I_1}{2\pi d},$$

and it points into the page. The Lorentz force law (in the form appropriate to line currents, Eq. 5.17) predicts a force directed towards (1), of magnitude

$$F = I_2\left(\frac{\mu_0 I_1}{2\pi d}\right)\int dl.$$

The *total* force, not surprisingly, is infinite, but the force per unit length is

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}. \tag{5.40}$$

If the currents are antiparallel (one up, one down), the force is repulsive—consistent again with the qualitative observations in Sect. 5.1.1.

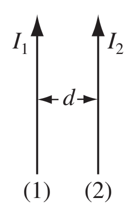


FIGURE 5.20

Example 5.6. Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I (Fig. 5.21).

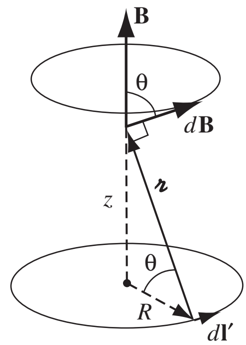


FIGURE 5.21

Solution

The field $d\mathbf{B}$ attributable to the segment $d\mathbf{l}'$ points as shown. As we integrate $d\mathbf{l}'$ around the loop, $d\mathbf{B}$ sweeps out a cone. The horizontal components cancel, and the vertical components combine, to give

$$B(z) = \frac{\mu_0}{4\pi} I \int \frac{dl'}{\mathscr{r}^2} \cos\theta.$$

(Notice that $d\mathbf{l}'$ and $\boldsymbol{\mathscr{r}}$ are perpendicular, in this case; the factor of $\cos\theta$ projects out the vertical component.) Now, $\cos\theta$ and $\mathscr{r}^2$ are constants, and $\int dl'$ is simply the circumference, $2\pi R$, so

$$B(z) = \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{\mathscr{r}^2}\right) 2\pi R = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}. \tag{5.41}$$

For surface and volume currents, the Biot-Savart law becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, da' \qquad \text{and} \qquad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau', \tag{5.42}$$

respectively. You might be tempted to write down the corresponding formula for a moving point charge, using the "dictionary" (Eq. 5.30):

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{q\mathbf{v} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}, \tag{5.43}$$

but this is simply *wrong*.[11] As I mentioned earlier, a point charge does not constitute a steady current, and the Biot-Savart law, which only holds for steady currents, does *not* correctly determine its field.

The superposition principle applies to magnetic fields just as it does to electric fields: if you have a *collection* of source currents, the net field is the (vector) sum of the fields due to each of them taken separately.

Problem 5.8

(a) Find the magnetic field at the center of a square loop, which carries a steady current I. Let R be the distance from center to side (Fig. 5.22).

(b) Find the field at the center of a regular n-sided polygon, carrying a steady current I. Again, let R be the distance from the center to any side.

(c) Check that your formula reduces to the field at the center of a circular loop, in the limit $n \to \infty$.

Problem 5.9 Find the magnetic field at point P for each of the steady current configurations shown in Fig. 5.23.

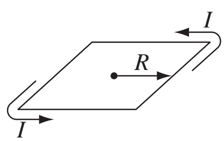
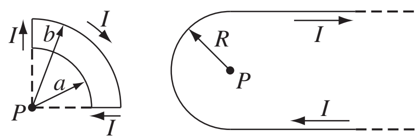


FIGURE 5.22



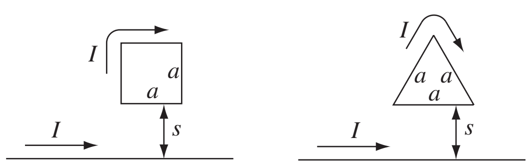
(a) (b)

FIGURE 5.23

Problem 5.10

(a) Find the force on a square loop placed as shown in Fig. 5.24(a), near an infinite straight wire. Both the loop and the wire carry a steady current I.

(b) Find the force on the triangular loop in Fig. 5.24(b).



(a) (b)

FIGURE 5.24

[11] I say this loud and clear to emphasize the point of principle; actually, Eq. 5.43 is *approximately* right for nonrelativistic charges ($v \ll c$), under conditions where retardation can be neglected (see Ex. 10.4).

Problem 5.11 Find the magnetic field at point P on the axis of a tightly wound **solenoid** (helical coil) consisting of n turns per unit length wrapped around a cylindrical tube of radius a and carrying current I (Fig. 5.25). Express your answer in terms of θ_1 and θ_2 (it's easiest that way). Consider the turns to be essentially circular, and use the result of Ex. 5.6. What is the field on the axis of an *infinite* solenoid (infinite in both directions)?

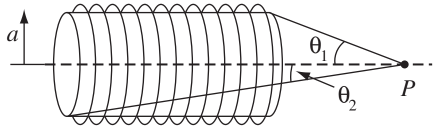


FIGURE 5.25

Problem 5.12 Use the result of Ex. 5.6 to calculate the magnetic field at the center of a uniformly charged spherical shell, of radius R and total charge Q, spinning at constant angular velocity ω.

Problem 5.13 Suppose you have two infinite straight line charges λ, a distance d apart, moving along at a constant speed v (Fig. 5.26). How great would v have to be in order for the magnetic attraction to balance the electrical repulsion? Work out the actual number. Is this a reasonable sort of speed?[12]

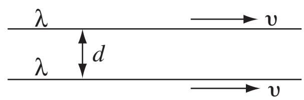


FIGURE 5.26

5.3 ■ THE DIVERGENCE AND CURL OF B

5.3.1 ■ Straight-Line Currents

The magnetic field of an infinite straight wire is shown in Fig. 5.27 (the current is coming *out* of the page). At a glance, it is clear that this field has a nonzero curl (something you'll never see in an *electro*static field); let's calculate it.

According to Eq. 5.38, the integral of **B** around a circular path of radius s, centered at the wire, is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s}\, dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I.$$

Notice that the answer is independent of s; that's because B *de*creases at the same rate as the circumference *in*creases. In fact, it doesn't have to be a circle; *any* old

[12]If you've studied special relativity, you may be tempted to look for complexities in this problem that are not really there—λ and v are both measured *in the laboratory frame*, and this is *ordinary electrostatics*.

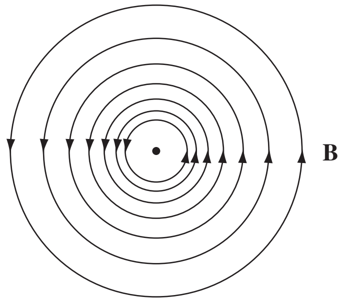


FIGURE 5.27

loop that encloses the wire would give the same answer. For if we use cylindrical coordinates (s, ϕ, z), with the current flowing along the z axis, $\mathbf{B} = (\mu_0 I/2\pi s)\hat{\boldsymbol{\phi}}$ and $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$, so

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s}\, s\, d\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I.$$

This assumes the loop encircles the wire exactly once; if it went around twice, then ϕ would run from 0 to 4π, and if it didn't enclose the wire at all, then ϕ would go from ϕ_1 to ϕ_2 and back again, with $\int d\phi = 0$ (Fig. 5.28).

Now suppose we have a *bundle* of straight wires. Each wire that passes through our loop contributes $\mu_0 I$, and those outside contribute nothing (Fig. 5.29). The line integral will then be

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}, \tag{5.44}$$

where I_{enc} stands for the total current enclosed by the integration path. If the flow of charge is represented by a volume current density $\mathbf{J}$, the enclosed current is

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}, \tag{5.45}$$

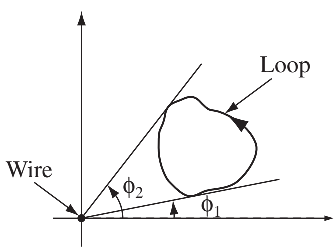


FIGURE 5.28

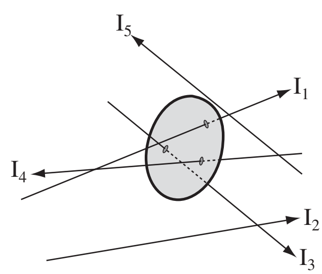


FIGURE 5.29

with the integral taken over any surface bounded by the loop. Applying Stokes' theorem to Eq. 5.44, then,

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a},$$

and hence

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{5.46}$$

With minimal labor, we have actually obtained the general formula for the curl of **B**. But our derivation is seriously flawed by the restriction to infinite straight line currents (and combinations thereof). Most current configurations *cannot* be constructed out of infinite straight wires, and we have no right to assume that Eq. 5.46 applies to them. So the next section is devoted to the formal derivation of the divergence and curl of **B**, starting from the Biot-Savart law itself.

5.3.2 ■ The Divergence and Curl of B

The Biot-Savart law for the general case of a volume current reads

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau'. \tag{5.47}$$

This formula gives the magnetic field at a point $\mathbf{r} = (x, y, z)$ in terms of an integral over the current distribution $\mathbf{J}(x', y', z')$ (Fig. 5.30). It is best to be absolutely explicit at this stage:

$$\mathbf{B} \text{ is a function of } (x, y, z),$$

$$\mathbf{J} \text{ is a function of } (x', y', z'),$$

$$\boldsymbol{\mathscr{r}} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

The integration is over the *primed* coordinates; the divergence and the curl of **B** are with respect to the *unprimed* coordinates.

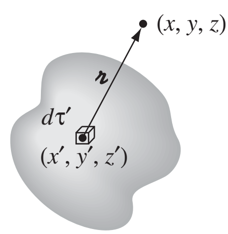


FIGURE 5.30

Applying the divergence to Eq. 5.47, we obtain:

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau'. \tag{5.48}$$

Invoking product rule number 6,

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right). \tag{5.49}$$

But $\nabla \times \mathbf{J} = 0$, because $\mathbf{J}$ doesn't depend on the unprimed variables, while $\nabla \times (\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2) = \mathbf{0}$ (Prob. 1.63), so

$$\boxed{\nabla \cdot \mathbf{B} = 0.} \tag{5.50}$$

Evidently the *divergence* of the magnetic field is *zero*.

Applying the curl to Eq. 5.47, we obtain:

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau'. \tag{5.51}$$

Again, our strategy is to expand the integrand, using the appropriate product rule—in this case number 8:

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{5.52}$$

(I have dropped terms involving derivatives of $\mathbf{J}$, because $\mathbf{J}$ does not depend on x, y, z.) The second term integrates to zero, as we'll see in the next paragraph. The first term involves the divergence we were at pains to calculate in Chapter 1 (Eq. 1.100):

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\boldsymbol{\mathscr{r}}). \tag{5.53}$$

Thus

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta^3(\mathbf{r} - \mathbf{r}')\, d\tau' = \mu_0 \mathbf{J}(\mathbf{r}),$$

which confirms that Eq. 5.46 is not restricted to straight-line currents, but holds quite generally in magnetostatics.

To complete the argument, however, we must check that the second term in Eq. 5.52 integrates to zero. Because the derivative acts only on $\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$, we can switch from ∇ to ∇' at the cost of a minus sign:[13]

$$-(\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = (\mathbf{J} \cdot \nabla') \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{5.54}$$

[13]The point here is that $\boldsymbol{\mathscr{r}}$ depends only on the *difference* between the coordinates; note that $(\partial/\partial x)\, f(x - x') = -(\partial/\partial x') f(x - x')$.

The x component, in particular, is

$$(\mathbf{J}\cdot\nabla')\left(\frac{x-x'}{\mathscr{r}^3}\right)=\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]-\left(\frac{x-x'}{\mathscr{r}^3}\right)(\nabla'\cdot\mathbf{J})$$

(using product rule 5). Now, for *steady* currents the divergence of **J** is zero (Eq. 5.33), so

$$\left[-(\mathbf{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]_x=\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right],$$

and therefore this contribution to the integral (Eq. 5.51) can be written

$$\int_{\mathcal{V}}\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]d\tau'=\oint_{\mathcal{S}}\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\cdot d\mathbf{a}'. \tag{5.55}$$

(The reason for switching from ∇ to ∇' was to permit this integration by parts.) But what region are we integrating over? Well, it's the volume that appears in the Biot-Savart law (Eq. 5.47)—large enough, that is, to include all the current. You can make it *bigger* than that, if you like; $\mathbf{J}=0$ out there anyway, so it will add nothing to the integral. The essential point is that *on the boundary* the current is *zero* (all current is safely *inside*) and hence the surface integral (Eq. 5.55) vanishes.[14]

5.3.3 ■ Ampère's Law

The equation for the curl of **B**,

$$\boxed{\nabla\times\mathbf{B}=\mu_0\mathbf{J},} \tag{5.56}$$

is called **Ampère's law** (in differential form). It can be converted to integral form by the usual device of applying one of the fundamental theorems—in this case Stokes' theorem:

$$\int(\nabla\times\mathbf{B})\cdot d\mathbf{a}=\oint\mathbf{B}\cdot d\mathbf{l}=\mu_0\int\mathbf{J}\cdot d\mathbf{a}.$$

Now, $\int\mathbf{J}\cdot d\mathbf{a}$ is the total current passing through the surface (Fig. 5.31), which we call I_{enc} (the **current enclosed** by the **Amperian loop**). Thus

$$\boxed{\oint\mathbf{B}\cdot d\mathbf{l}=\mu_0 I_{\text{enc}}.} \tag{5.57}$$

[14]If **J** itself extends to infinity (as in the case of an infinite straight wire), the surface integral is still typically zero, though the analysis calls for greater care.

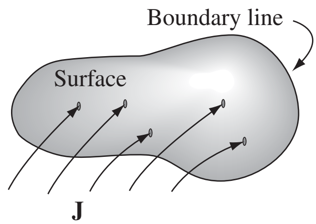


FIGURE 5.31

This is the integral version of Ampère's law; it generalizes Eq. 5.44 to *arbitrary* steady currents. Notice that Eq. 5.57 inherits the sign ambiguity of Stokes' theorem (Sect. 1.3.5): Which *way* around the loop am I supposed to go? And which *direction* through the surface corresponds to a "positive" current? The resolution, as always, is the right-hand rule: If the fingers of your right hand indicate the direction of integration around the boundary, then your thumb defines the direction of a positive current.

Just as the Biot-Savart law plays a role in magnetostatics that Coulomb's law assumed in electrostatics, so Ampère's plays the part of Gauss's:

$$\begin{cases} \text{Electrostatics :} & \text{Coulomb} & \rightarrow & \text{Gauss,} \\ \text{Magnetostatics :} & \text{Biot}-\text{Savart} & \rightarrow & \text{Ampère.} \end{cases}$$

In particular, for currents with appropriate symmetry, Ampère's law in integral form offers a lovely and extraordinarily efficient way of calculating the magnetic field.

Example 5.7. Find the magnetic field a distance s from a long straight wire (Fig. 5.32), carrying a steady current I (the same problem we solved in Ex. 5.5, using the Biot-Savart law).

Solution

We know the direction of $\mathbf{B}$ is "circumferential," circling around the wire as indicated by the right-hand rule. By symmetry, the magnitude of $\mathbf{B}$ is constant around an Amperian loop of radius s, centered on the wire. So Ampère's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint dl = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I,$$

or

$$B = \frac{\mu_0 I}{2\pi s}.$$

This is the same answer we got before (Eq. 5.38), but it was obtained this time with far less effort.

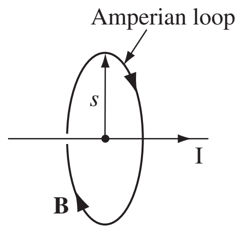


FIGURE 5.32

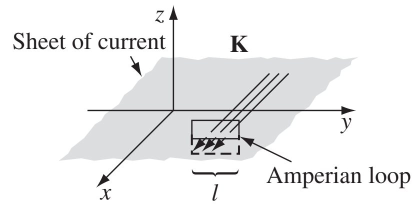


FIGURE 5.33

Example 5.8. Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K\,\hat{\mathbf{x}}$, flowing over the xy plane (Fig. 5.33).

Solution

First of all, what is the *direction* of **B**? Could it have any x component? *No*: A glance at the Biot-Savart law (Eq. 5.42) reveals that **B** is *perpendicular* to **K**. Could it have a z component? *No again*. You could confirm this by noting that any vertical contribution from a filament at $+y$ is canceled by the corresponding filament at $-y$. But there is a nicer argument: Suppose the field pointed *away* from the plane. By reversing the direction of the current, I could make it point *toward* the plane (in the Biot-Savart law, changing the sign of the current switches the sign of the field). But the z component of **B** cannot possibly depend on the *direction* of the current in the xy plane. (Think about it!) So **B** can only have a y component, and a quick check with your right hand should convince you that it points to the *left* above the plane and to the *right* below it.

With this in mind, we draw a rectangular Amperian loop as shown in Fig. 5.33, parallel to the yz plane and extending an equal distance above and below the surface. Applying Ampère's law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = 2Bl = \mu_0 I_{\text{enc}} = \mu_0 K l,$$

(one Bl comes from the top segment and the other from the bottom), so $B = (\mu_0/2)K$, or, more precisely,

$$\mathbf{B} = \begin{cases} +(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z < 0, \\ -(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z > 0. \end{cases} \tag{5.58}$$

Notice that the field is independent of the distance from the plane, just like the electric field of a uniform surface charge (Ex. 2.5).

Example 5.9. Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R, each carrying a steady current I (Fig. 5.34). [The point of making the windings so close is that one can then pretend each turn is circular. If this troubles you (after all, there is a net current I in the direction of the solenoid's axis, no matter *how* tight the

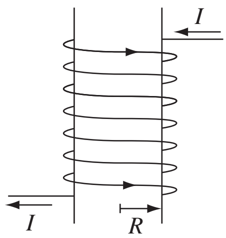


FIGURE 5.34

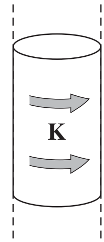


FIGURE 5.35

winding), picture instead a sheet of aluminum foil wrapped around the cylinder, carrying the equivalent uniform surface current $K = nI$ (Fig. 5.35). Or make a double winding, going up to one end and then—always in the same sense—going back down again, thereby eliminating the net longitudinal current. But, in truth, this is all unnecessary fastidiousness, for the field inside a solenoid is huge (relatively speaking), and the field of the longitudinal current is at most a tiny refinement.]

Solution

First of all, what is the *direction* of **B**? Could it have a radial component? *No*. For suppose B_s were *positive*; if we reversed the direction of the current, B_s would then be *negative*. But switching I is physically equivalent to turning the solenoid upside down, and that certainly should not alter the radial field. How about a "circumferential" component? *No*. For B_ϕ would be constant around an Amperian loop concentric with the solenoid (Fig. 5.36), and hence

$$\oint \mathbf{B} \cdot d\mathbf{l} = B_\phi(2\pi s) = \mu_0 I_{\text{enc}} = 0,$$

since the loop encloses no current.

So the magnetic field of an infinite, closely wound solenoid runs *parallel to the axis*. From the right-hand rule, we expect that it points upward inside the solenoid and downward outside. Moreover, it certainly approaches zero as you go very far

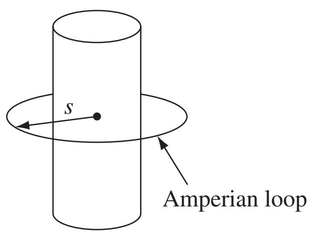


FIGURE 5.36

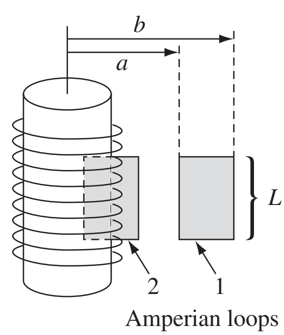


FIGURE 5.37

away. With this in mind, let's apply Ampère's law to the two rectangular loops in Fig. 5.37. Loop 1 lies entirely outside the solenoid, with its sides at distances a and b from the axis:

$$\oint \mathbf{B} \cdot d\mathbf{l} = [B(a) - B(b)]L = \mu_0 I_{\text{enc}} = 0,$$

so

$$B(a) = B(b).$$

Evidently the *field outside does not depend on the distance from the axis*. But we agreed that it goes to *zero* for large s. It must therefore be zero *everywhere*! (This astonishing result can also be derived from the Biot-Savart law, of course, but it's much more difficult. See Prob. 5.46.)

As for loop 2, which is half inside and half outside, Ampère's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 I_{\text{enc}} = \mu_0 n I L,$$

where B is the field inside the solenoid. (The right side of the loop contributes nothing, since $B = 0$ out there.) *Conclusion:*

$$\mathbf{B} = \begin{cases} \mu_0 n I\, \hat{\mathbf{z}}, & \text{inside the solenoid,} \\ \mathbf{0}, & \text{outside the solenoid.} \end{cases} \tag{5.59}$$

Notice that the field inside is *uniform*—it doesn't depend on the distance from the axis. In this sense the solenoid is to magnetostatics what the parallel-plate capacitor is to electrostatics: a simple device for producing strong uniform fields.

Like Gauss's law, Ampère's law is always *true* (for steady currents), but it is not always *useful*. Only when the symmetry of the problem enables you to pull B outside the integral $\oint \mathbf{B} \cdot d\mathbf{l}$ can you calculate the magnetic field from Ampère's law. When it *does* work, it's by far the fastest method; when it doesn't, you have to fall back on the Biot-Savart law. The current configurations that can be handled by Ampère's law are

1. Infinite straight lines (prototype: Ex. 5.7).
2. Infinite planes (prototype: Ex. 5.8).
3. Infinite solenoids (prototype: Ex. 5.9).
4. Toroids (prototype: Ex. 5.10).

The last of these is a surprising and elegant application of Ampère's law. As in Exs. 5.8 and 5.9, the hard part is figuring out the *direction* of the field (which we will now have done, once and for all, for each of the four geometries); the actual application of Ampère's law takes only one line.

Example 5.10. A toroidal coil consists of a circular ring, or "donut," around which a long wire is wrapped (Fig. 5.38). The winding is uniform and tight enough so that each turn can be considered a plane closed loop. The cross-sectional shape of the coil is immaterial. I made it rectangular in Fig. 5.38 for the sake of simplicity, but it could just as well be circular or even some weird asymmetrical form, as in Fig. 5.39, as long as the shape remains the same all the way around the ring. In that case, it follows that the *magnetic field of the toroid is circumferential at all points, both inside and outside the coil*.

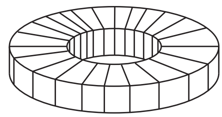


FIGURE 5.38

Proof. According to the Biot-Savart law, the field at $\mathbf{r}$ due to the current element at $\mathbf{r}'$ is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{I}\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}\,dl'.$$

We may as well put $\mathbf{r}$ in the xz plane (Fig. 5.39), so its Cartesian components are $(x, 0, z)$, while the source coordinates are

$$\mathbf{r}' = (s'\cos\phi', s'\sin\phi', z').$$

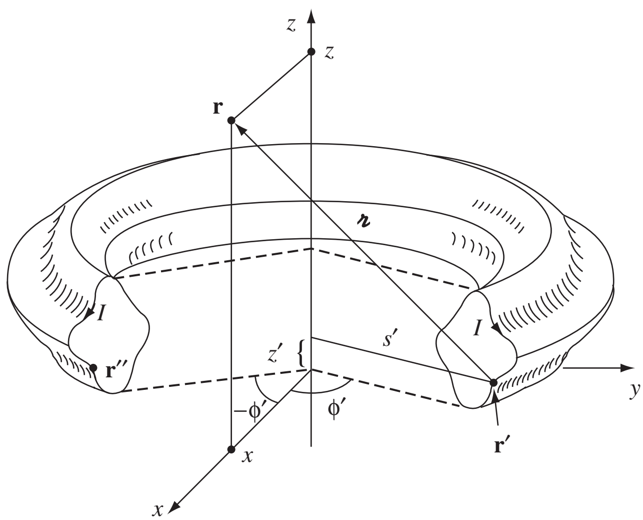


FIGURE 5.39

Then

$$\boldsymbol{\mathscr{r}} = (x - s'\cos\phi', -s'\sin\phi', z - z').$$

Since the current has no ϕ component, $\mathbf{I} = I_s\,\hat{\mathbf{s}} + I_z\,\hat{\mathbf{z}}$, or (in Cartesian coordinates)

$$\mathbf{I} = (I_s\cos\phi', I_s\sin\phi', I_z).$$

Accordingly,

$$\begin{aligned}\mathbf{I} \times \boldsymbol{\mathscr{r}} &= \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ I_s\cos\phi' & I_s\sin\phi' & I_z \\ (x - s'\cos\phi') & (-s'\sin\phi') & (z - z') \end{bmatrix} \\ &= \left[\sin\phi'\left(I_s(z - z') + s'I_z\right)\right]\hat{\mathbf{x}} + \left[I_z(x - s'\cos\phi') - I_s\cos\phi'(z - z')\right]\hat{\mathbf{y}} \\ &\quad + \left[-I_s x\sin\phi'\right]\hat{\mathbf{z}}.\end{aligned}$$

But there is a symmetrically situated current element at $\mathbf{r}''$, with the same s', the same $\mathscr{r}$, the same dl', the same I_s, and the same I_z, *but negative* ϕ' (Fig. 5.39). Because $\sin\phi'$ changes sign, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ contributions from $\mathbf{r}'$ and $\mathbf{r}''$ cancel, leaving only a $\hat{\mathbf{y}}$ term. Thus the field at $\mathbf{r}$ is in the $\hat{\mathbf{y}}$ direction, and in general the field points in the $\hat{\boldsymbol{\phi}}$ direction. □

Now that we know the field is circumferential, determining its magnitude is ridiculously easy. Just apply Ampère's law to a circle of radius s about the axis of the toroid:

$$B2\pi s = \mu_0 I_{\text{enc}},$$

and hence

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{for points inside the coil,} \\[2ex] \mathbf{0}, & \text{for points outside the coil,} \end{cases} \tag{5.60}$$

where N is the total number of turns.

Problem 5.14 A steady current I flows down a long cylindrical wire of radius a (Fig. 5.40). Find the magnetic field, both inside and outside the wire, if

(a) The current is uniformly distributed over the outside surface of the wire.

(b) The current is distributed in such a way that J is proportional to s, the distance from the axis.

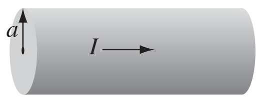


FIGURE 5.40

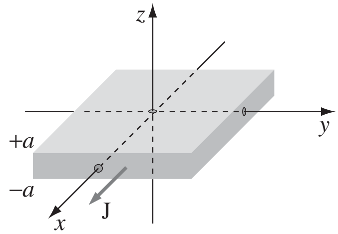


FIGURE 5.41

Problem 5.15 A thick slab extending from $z = -a$ to $z = +a$ (and infinite in the x and y directions) carries a uniform volume current $\mathbf{J} = J\,\hat{\mathbf{x}}$ (Fig. 5.41). Find the magnetic field, as a function of z, both inside and outside the slab.

Problem 5.16 Two long coaxial solenoids each carry current I, but in opposite directions, as shown in Fig. 5.42. The inner solenoid (radius a) has n_1 turns per unit length, and the outer one (radius b) has n_2. Find $\mathbf{B}$ in each of the three regions: (i) inside the inner solenoid, (ii) between them, and (iii) outside both.

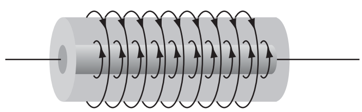


FIGURE 5.42

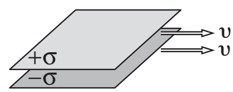


FIGURE 5.43

Problem 5.17 A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in Fig. 5.43.

(a) Find the magnetic field between the plates and also above and below them.

(b) Find the magnetic force per unit area on the upper plate, including its direction.

(c) At what speed v would the magnetic force balance the electrical force?[15]

! **Problem 5.18** Show that the magnetic field of an infinite solenoid runs parallel to the axis, *regardless of the cross-sectional shape of the coil*, as long as that shape is constant along the length of the solenoid. What is the magnitude of the field, inside and outside of such a coil? Show that the toroid field (Eq. 5.60) reduces to the solenoid field, when the radius of the donut is so large that a segment can be considered essentially straight.

Problem 5.19 In calculating the current enclosed by an Amperian loop, one must, in general, evaluate an integral of the form

$$I_{\text{enc}} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}.$$

[15] See footnote to Prob. 5.13.

The trouble is, there are infinitely many surfaces that share the same boundary line. Which one are we supposed to use?

5.3.4 ■ Comparison of Magnetostatics and Electrostatics

The divergence and curl of the *electrostatic* field are

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(Gauss's law);} \\ \\ \nabla \times \mathbf{E} = \mathbf{0}, & \text{(no name).} \end{cases}$$

These are **Maxwell's equations** for electrostatics. Together with the boundary condition $\mathbf{E} \to \mathbf{0}$ far from all charges,[16] Maxwell's equations determine the field, if the source charge density ρ is given; they contain essentially the same information as Coulomb's law plus the principle of superposition. The divergence and curl of the *magnetostatic* field are

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

These are Maxwell's equations for magnetostatics. Again, together with the boundary condition $\mathbf{B} \to \mathbf{0}$ far from all currents, Maxwell's equations determine the magnetic field; they are equivalent to the Biot-Savart law (plus superposition). Maxwell's equations and the force law

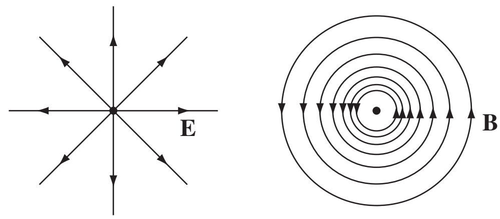
$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

constitute the most elegant formulation of electrostatics and magnetostatics.

The electric field *diverges away from* a (positive) charge; the magnetic field line *curls around* a current (Fig. 5.44). Electric field lines originate on positive charges and terminate on negative ones; magnetic field lines do not begin or end anywhere—to do so would require a nonzero divergence. They typically form closed loops or extend out to infinity.[17] To put it another way, *there are no point sources for* **B**, as there are for **E**; there exists no magnetic analog to electric charge. This is the physical content of the statement $\nabla \cdot \mathbf{B} = 0$. Coulomb and others believed that magnetism was produced by **magnetic charges** (**magnetic monopoles**, as we would now call them), and in some older books you will still find references to a magnetic version of Coulomb's law, giving the force of attraction or repulsion between them. It was Ampère who first speculated that all magnetic effects are attributable to *electric* charges *in motion* (currents). As far

[16] In those artificial problems where the charge (or current) extends to infinity—infinite planes, for example—symmetry considerations can sometimes take the place of boundary conditions.

[17] A third possibility turns out to be surprisingly common: they can form chaotic tangles. See M. Lieberherr, *Am. J. Phys.* **78**, 1117 (2010).



(a) Electrostatic field of a point charge

(b) Magnetostatic field of a long wire

FIGURE 5.44

as we know, Ampère was right; nevertheless, it remains an open experimental question whether magnetic monopoles exist in nature (they are obviously pretty *rare*, or somebody would have found one[18]), and in fact some recent elementary particle theories *require* them. For our purposes, though, **B** is divergenceless, and there are no magnetic monopoles. It takes a *moving* electric charge to *produce* a magnetic field, and it takes another moving electric charge to "feel" a magnetic field.

Typically, electric forces are enormously larger than magnetic ones. That's not something intrinsic to the theory; it has to do with the sizes of the fundamental constants ϵ_0 and μ_0. In general, it is only when both the source charges and the test charge are moving at velocities comparable to the speed of light that the magnetic force approaches the electric force in strength. (Problems 5.13 and 5.17 illustrate this rule.) How is it, then, that we notice magnetic effects at all? The answer is that both in the production of a magnetic field (Biot-Savart) and in its detection (Lorentz), it is the *current* that matters, and we can compensate for a smallish velocity by pouring huge amounts of charge down the wire. Ordinarily, this charge would simultaneously generate so large an *electric* force as to swamp the magnetic one. But if we arrange to keep the wire *neutral*, by embedding in it an equal quantity of opposite charge at rest, the electric field cancels out, leaving the magnetic field to stand alone. It sounds very elaborate, but of course this is precisely what happens in an ordinary current carrying wire.

Problem 5.20

(a) Find the density ρ of mobile charges in a piece of copper, assuming each atom contributes one free electron. [Look up the necessary physical constants.]

(b) Calculate the average electron velocity in a copper wire 1 mm in diameter, carrying a current of 1 A. [*Note*: This is literally a *snail's* pace. How, then, can you carry on a long distance telephone conversation?]

[18] An apparent detection (B. Cabrera, *Phys. Rev. Lett.* **48**, 1378 (1982)) has never been reproduced—and not for want of trying. For a delightful brief history of ideas about magnetism, see Chapter 1 in D. C. Mattis, *The Theory of Magnetism* (New York: Harper & Row, 1965).

(c) What is the force of attraction between two such wires, 1 cm apart?

(d) If you could somehow remove the stationary positive charges, what would the electrical repulsion force be? How many times greater than the magnetic force is it?

Problem 5.21 Is Ampère's law consistent with the general rule (Eq. 1.46) that divergence-of-curl is always zero? Show that Ampère's law *cannot* be valid, in general, outside magnetostatics. Is there any such "defect" in the other three Maxwell equations?

Problem 5.22 Suppose there *did* exist magnetic monopoles. How would you modify Maxwell's equations and the force law to accommodate them? If you think there are several plausible options, list them, and suggest how you might decide experimentally which one is right.

6

The magnetic field

Overview Having shown in Chapter 5 that the magnetic force must exist, we will now study the various properties of the magnetic field and show how it can be calculated for an arbitrary (steady) current distribution. The *Lorentz force* gives the total force on a charged particle as $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$. The results from the previous chapter give us the form of the magnetic field due to a long straight wire. This form leads to *Ampère's law*, which relates the line integral of the magnetic field to the current enclosed by the integration loop. It turns out that Ampère's law holds for a wire of any shape. When supplemented with a term involving changing electric fields, this law becomes one of Maxwell's equations (as we will see in Chapter 9). The sources of magnetic fields are currents, in contrast with the sources of electric fields, which are charges; there are no isolated magnetic charges, or *monopoles*. This statement is another of Maxwell's equations.

As in the electric case, the magnetic field can be obtained from a potential, but it is now a *vector potential*; its curl gives the magnetic field. The *Biot–Savart law* allows us to calculate (in principle) the magnetic field due to any steady current distribution. One distribution that comes up often is that of a *solenoid* (a coil of wire), whose field is (essentially) constant inside and zero outside. This field is consistent with an Ampère's-law calculation of the discontinuity of $\mathbf{B}$ across a sheet of current. By considering various special cases, we derive the *Lorentz transformations* of the electric and magnetic fields. The electric (or magnetic) field in one frame depends on *both* the electric and magnetic fields in another frame. The *Hall effect* arises from the $q\mathbf{v} \times \mathbf{B}$ part of the

Lorentz force. This effect allows us, for the first time, to determine the sign of the charge carriers in a current.

6.1 Definition of the magnetic field

A charge that is moving parallel to a current of other charges experiences a force perpendicular to its own velocity. We can see it happening in the deflection of the electron beam in Fig. 5.3. We discovered in Section 5.9 that this is consistent with – indeed, is required by – Coulomb's law combined with charge invariance and special relativity. And we found that a force perpendicular to the charged particle's velocity also arises in motion at right angles to the current-carrying wire. For a given current, the magnitude of the force, which we calculated for the particular case in Fig. 5.22(a), is proportional to the product of the particle's charge q and its speed v in our frame. Just as we defined the electric field $\mathbf{E}$ as the vector force on unit charge at rest, so we can define another field $\mathbf{B}$ by the *velocity-dependent* part of the force that acts on a charge in motion. The defining relation was introduced at the beginning of Chapter 5. Let us state it again more carefully.

At some instant t a particle of charge q passes the point (x, y, z) in our frame, moving with velocity $\mathbf{v}$. At that moment the force on the particle (its rate of change of momentum) is $\mathbf{F}$. The electric field at that time and place is known to be $\mathbf{E}$. Then the magnetic field at that time and place is defined as the vector $\mathbf{B}$ that satisfies the following vector equation (for any value of $\mathbf{v}$):

$$\boxed{\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}} \tag{6.1}$$

This force $\mathbf{F}$ is called the *Lorentz force*. Of course, $\mathbf{F}$ here includes only the charge-dependent force and not, for instance, the weight of the particle carrying the charge. A vector $\mathbf{B}$ satisfying Eq. (6.1) always exists. Given the values of $\mathbf{E}$ and $\mathbf{B}$ in some region, we can with Eq. (6.1) predict the force on any particle moving through that region with any velocity. For fields that vary in time and space, Eq. (6.1) is to be understood as a local relation among the instantaneous values of $\mathbf{F}$, $\mathbf{E}$, $\mathbf{v}$, and $\mathbf{B}$. Of course, all four of these quantities must be measured in the same inertial frame.

In the case of our "test charge" in the lab frame of Fig. 5.22(a), the electric field $\mathbf{E}$ was zero. With the charge q moving in the positive x direction, $\mathbf{v} = \hat{\mathbf{x}}v$, we found in Eq. (5.28) that the force on it was in the negative y direction, with magnitude $Iqv/2\pi\epsilon_0 rc^2$:

$$\mathbf{F} = -\hat{\mathbf{y}}\frac{Iqv}{2\pi\epsilon_0 rc^2}. \tag{6.2}$$

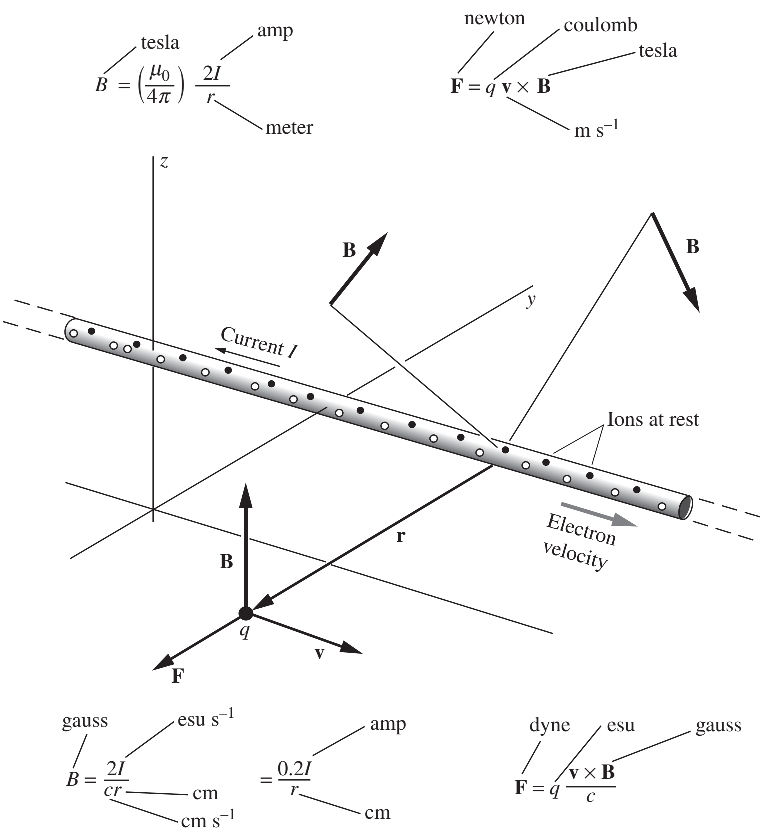


Figure 6.1.
The magnetic field of a current in a long straight wire and the force on a charged particle moving through that field.

In this case the magnetic field must be

$$\boxed{\mathbf{B} = \hat{\mathbf{z}}\frac{I}{2\pi\epsilon_0 rc^2}} \tag{6.3}$$

for then Eq. (6.1) becomes

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B} = (\hat{\mathbf{x}}\times\hat{\mathbf{z}})(qv)\left(\frac{I}{2\pi\epsilon_0 rc^2}\right) = -\hat{\mathbf{y}}\frac{Iqv}{2\pi\epsilon_0 rc^2}, \tag{6.4}$$

in agreement with Eq. (6.2).

The relation of **B** to **r** and to the current I is shown in Fig. 6.1. Three mutually perpendicular directions are involved: the direction of **B** at the point of interest, the direction of a vector **r** from the wire to that point, and the direction of current flow in the wire. Here questions of *handedness* arise for the first time in our study. Having adopted Eq. (6.1) as the definition of **B** and agreed on the conventional rule for the vector product, that is, $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$, etc., in coordinates like those of Fig. 6.1, we have determined the direction of **B**. That relation has a

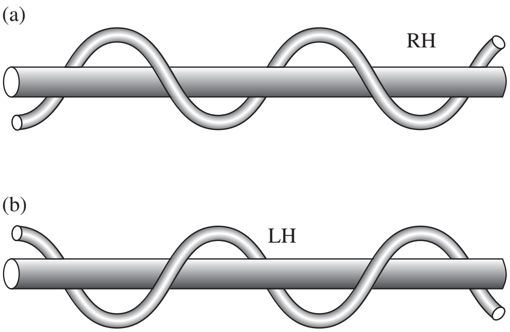


Figure 6.2.
A reminder. The helix in (a) is called a right-handed helix, that in (b) a left-handed helix.

handedness, as you can see by imagining a particle that moves along the wire in the direction of the current while circling around the wire in the direction of **B**. Its trail, no matter how you look at it, would form a right-hand helix, like that in Fig. 6.2(a), not a left-hand helix like that in Fig. 6.2(b).

From the $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ relation, we see that another set of three (not necessarily mutually perpendicular) vectors consists of the force **F** on the charge q, the velocity **v** of the charge, and the magnetic field **B** at the location of the charge. In Fig. 6.1, **v** happens to point along the direction of the wire, and **F** along the direction of **r**, but these need not be the directions in general; **F** will always be perpendicular to both **v** and **B**, but **v** can point in any direction of your choosing, so it need not be perpendicular to **B**.

Consider an experiment like Oersted's, as pictured in Fig. 5.2(a). The direction of the current was settled when the wire was connected to the battery. Which way the compass needle points can be stated if we color one end of the needle and call it the head of the arrow. By tradition, long antedating Oersted, the "north-seeking" end of the needle is so designated, and that is the black end of the needle in Fig. 5.2(a).[1] If you compare that picture with Fig. 6.1, you will see that we have defined **B** so that it points in the direction of "local magnetic north." Or, to put it another way, the current arrow and the compass needle in Fig. 5.2(a) define a right-handed helix (see Fig. 6.2), as do the current direction and the vector **B** in Fig. 6.1. This is not to say that there is anything intrinsically right-handed about electromagnetism. It is only the self-consistency of our rules and definitions that concerns us here. Let us note, however, that a question of handedness *could never arise* in electrostatics. In this sense the vector **B** differs in character from the vector **E**. In the same way, a vector representing an angular velocity, in mechanics, differs from a vector representing a linear velocity.

The SI units of **B** can be determined from Eq. (6.1). In a magnetic field of unit strength, a charge of one coulomb moving with a velocity of one meter/second perpendicular to the field experiences a force of one newton. The unit of **B** so defined is called the *tesla*:

$$1\ \text{tesla} = 1\ \frac{\text{newton}}{\text{coulomb} \cdot \text{meter/second}} = 1\ \frac{\text{newton}}{\text{amp} \cdot \text{meter}}. \tag{6.5}$$

In terms of other units, 1 tesla equals $1\,\text{kg}\,\text{C}^{-1}\,\text{s}^{-1}$. In SI units, the relation between field and current in Eq. (6.3) is commonly written as

$$\boxed{\mathbf{B} = \hat{\mathbf{z}}\frac{\mu_0 I}{2\pi r}} \tag{6.6}$$

[1] We now know that the earth's magnetic field has reversed many times in geologic history. See Problem 7.19 and the reference there given.

where **B** is in teslas, I is in amps, and r is in meters. The constant μ_0, like the constant ϵ_0 we met in electrostatics, is a fundamental constant in the SI unit system. Its value is defined to be exactly

$$\boxed{\mu_0 \equiv 4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}} \tag{6.7}$$

Of course, if Eq. (6.6) is to agree with Eq. (6.3), we must have

$$\mu_0 = \frac{1}{\epsilon_0 c^2} \quad \Longrightarrow \quad \boxed{c^2 = \frac{1}{\mu_0\epsilon_0}} \tag{6.8}$$

With ϵ_0 given in Eq. (1.3), and $c = 2.998 \cdot 10^8$ m/s, you can quickly check that this relation does indeed hold.

REMARK: Given that we already found the **B** field due to a current-carrying wire in Eq. (6.3), you might wonder what the point is of rewriting **B** in terms of the newly introduced constant μ_0 in Eq. (6.6). The answer is that μ_0 is a product of the historical development of magnetism, which should be contrasted with the special-relativistic development we followed in Chapter 5. The connection between electric and magnetic effects was certainly observed long before the formulation of special relativity in 1905. In particular, as we learned in Section 5.1, Oersted discovered in 1820 that a current-carrying wire produces a magnetic field. And μ_0 was eventually introduced as the constant of proportionality in Eq. (6.6). (Or, more accurately, μ_0 was assigned a given value, and then Eq. (6.6) was used to define the unit of current.) But even with the observed connection between electricity and magnetism, in the mid nineteenth century there was no obvious relation between the μ_0 in the expression for **B** and the ϵ_0 in the expression for **E**. They were two separate constants in two separate theories. But two developments changed this.

First, in 1861 Maxwell wrote down his set of equations that govern all of electromagnetism. He then used these equations to show that electromagnetic waves exist and travel with speed $1/\sqrt{\mu_0\epsilon_0} \approx 3 \cdot 10^8$ m/s. (We'll study Maxwell's equations and electromagnetic waves in Chapter 9.) This strongly suggested that light is an electromagnetic wave, a fact that was demonstrated experimentally by Hertz in 1888. Therefore, $c = 1/\sqrt{\mu_0\epsilon_0}$, and hence $\mu_0 = 1/\epsilon_0 c^2$. This line of reasoning shows that the speed of light c is determined by the two constants ϵ_0 and μ_0.

The second development was Einstein's formulation of the special theory of relativity in 1905. Relativity was the basis of our reasoning in Chapter 5 (the main ingredients of which were length contraction and the relativistic velocity-addition formula), which led to the expression for the magnetic field in Eq. (6.3). A comparison of this equation with the historical expression in Eq. (6.6) yields $\mu_0 = 1/\epsilon_0 c^2$. This line of reasoning shows that μ_0 is determined by the two constants ϵ_0 and c. Of course, having proceeded the way we did in Chapter 5, there is no need to introduce the constant μ_0 in Eq. (6.6) when we already have Eq. (6.3). Nevertheless, the convention in SI units is to write **B** in the form given in Eq. (6.6). If you wish, you can think of μ_0 simply as a convenient shorthand for the more cumbersome expression $1/\epsilon_0 c^2$.

Comparing the previous two paragraphs, it is unclear which derivation of $\mu_0 = 1/\epsilon_0 c^2$ is "better." Is it preferable to take ϵ_0 and μ_0 as the fundamental constants and then derive, with Maxwell's help, the value of c, or to take ϵ_0 and

c as the fundamental constants and derive, with Einstein's help, the value of μ_0? The former derivation has the advantage of explaining why c takes on the value $2.998 \cdot 10^8$ m/s, while the latter has the advantage of explaining how magnetic forces arise from electric forces. In the end, it's a matter of opinion, based on what information you want to start with.

In Gaussian units, Eq. (6.1) takes the slightly different form

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B}. \tag{6.9}$$

Note that $\mathbf{B}$ now has the same dimensions as $\mathbf{E}$, the factor $\mathbf{v}/c$ being dimensionless. With force F in dynes and charge q in esu, the unit of magnetic field strength is the dyne/esu. This unit has a name, the *gauss*. There is no special name for the unit dyne/esu when it is used as a unit of electric field strength. It is the same as 1 statvolt/cm, which is the term normally used for unit electric field strength in the Gaussian system. In Gaussian units, the equation analogous to Eq. (6.3) is

$$\mathbf{B} = \hat{\mathbf{z}}\frac{2I}{rc}. \tag{6.10}$$

If you repeat the reasoning of Chapter 5, you will see that this $\mathbf{B}$ is obtained basically by replacing ϵ_0 by $1/4\pi$ and erasing one of the factors of c in Eq. (6.3). $\mathbf{B}$ is in gauss if I is in esu/s, r is in cm, and c is in cm/s.

Example (Relation between 1 tesla and 1 gauss) Show that 1 tesla is equivalent to exactly 10^4 gauss.

Solution Consider a setup where a charge of 1 C travels at 1 m/s in a direction perpendicular to a magnetic field with strength 1 tesla. Equations (6.1) and (6.5) tell us that the charge experiences a force of 1 newton. Let us express this fact in terms of the Gaussian force relation in Eq. (6.9). We know that $1\,\text{N} = 10^5$ dyne and $1\,\text{C} = 3 \cdot 10^9$ esu (this "3" isn't actually a 3; see the discussion below). If we let $1\,\text{tesla} = n$ gauss, with n to be determined, then the way that Eq. (6.9) describes the given situation is as follows:

$$10^5\ \text{dyne} = \frac{3 \cdot 10^9\ \text{esu}}{3 \cdot 10^{10}\ \text{cm/s}}\left(100\ \frac{\text{cm}}{\text{s}}\right)(n\ \text{gauss}). \tag{6.11}$$

Since 1 gauss equals 1 dyne/esu, all the units cancel, and we end up with $n = 10^4$, as desired.

Now, the two 3's in Eq. (6.11) are actually 2.998's. This is clear in the denominator because the 3 comes from the factor of c. To see why it is the case in the numerator, recall the example in Section 1.4 where we showed that $1\,\text{C} = 3 \cdot 10^9$ esu. If you redo that example and keep things in terms of the constant k given in Eqs. (1.2) and (1.3), you will find that the number $3 \cdot 10^9$ is actually $\sqrt{10^9 k}$ (ignoring the units of k). But in view of the definition of μ_0 in Eq. (6.7), the $k = 1/4\pi\epsilon_0$ expression in Eq. (1.3) can be written as $k = 1/(10^7\mu_0\epsilon_0)$. And we know from above that $1/\mu_0\epsilon_0 = c^2$, hence $k = 10^{-7}c^2$ (ignoring the units).

So the number $3 \cdot 10^9$ is really $\sqrt{10^9 k} = \sqrt{10^2 c^2} = 10c$ (ignoring the units), or $2.998 \cdot 10^9$. Since both of the 3's in Eq. (6.11) are modified in the same way, the $n = 10^4$ result is therefore still exact.

Let us use Eqs. (6.1) and (6.6) to calculate the magnetic force between parallel wires carrying current. Let r be the distance between the wires, and let I_1 and I_2 be the currents which we assume are flowing in the same direction, as shown in Fig. 6.3. The wires are assumed to be infinitely long – a fair assumption in a practical case if they are very long compared with the distance r between them. We want to predict the force that acts on some finite length l of one of the wires, due to the entirety of the other wire. The current in wire 1 causes a magnetic field of strength

$$B_1 = \frac{\mu_0 I_1}{2\pi r} \tag{6.12}$$

at the location of wire 2. Within wire 2 there are n_2 moving charges per meter length of wire, each with charge q_2 and speed v_2. They constitute the current I_2:

$$I_2 = n_2 q_2 v_2. \tag{6.13}$$

According to Eq. (6.1), the force on each charge is $q_2 v_2 B_1$.[2] The force on each meter length of wire is therefore $n_2 q_2 v_2 B_1$, or simply $I_2 B_1$. The force on a length l of wire 2 is then

$$\boxed{F = I_2 B_1 l} \tag{6.14}$$

Using the B_1 from Eq. (6.12), this becomes

$$\boxed{F = \frac{\mu_0 I_1 I_2 l}{2\pi r}} \tag{6.15}$$

Here F is in newtons, and I_1 and I_2 are in amps. As the factor l/r that appears both in Eq. (6.15) and below in Eq. (6.16) is dimensionless, l and r could be in any units.[3]

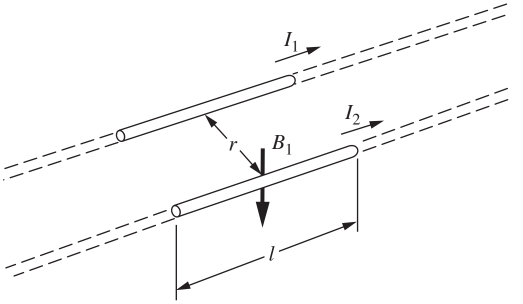


Figure 6.3.
Current I_1 produces magnetic field B_1 at conductor 2. The force on a length l of conductor 2 is given by Eq. (6.15).

[2] B_1 is the field *inside* wire 2, caused by the current in wire 1. When we study magnetic fields inside matter in Chapter 11, we will find that most conductors, including copper and aluminum, but *not* including iron, have very little influence on a magnetic field. For the present, let us agree to avoid things like iron and other ferromagnetic materials. Then we can safely assume that the magnetic field inside the wire is practically what it would be in vacuum with the same currents flowing.

[3] Equation (6.15) has usually been regarded as the primary definition of the ampere in the SI system, μ_0 being *assigned* the value $4\pi \cdot 10^{-7}$. That is to say, one ampere is the current that, flowing in each of two infinitely long parallel wires a distance r apart, will cause a force of exactly $2 \cdot 10^{-7}$ newton on a length $l = r$ of one of the wires. The other SI electrical units are then defined in terms of the ampere. Thus a coulomb is one ampere-second, a volt is one joule/coulomb, and an ohm is one volt/ampere.

The same exercise carried out in Gaussian units, with Eqs. (6.9) and (6.10), will lead to

$$F = \frac{2I_1 I_2 l}{c^2 r}. \tag{6.16}$$

Equation (6.15) is symmetric in the labels 1 and 2, so the force on an equal length of wire 1 caused by the field of wire 2 must be given by the same formula. We have not bothered to keep track of signs because we know already that currents in the same direction attract one another.

More generally, we can calculate the force on a small piece of current-carrying wire that sits in a magnetic field **B**. Let the length of the small piece be dl, the linear charge density of the moving charges be λ, and the speed of these charges be v. Then the amount of moving charge in the piece is $dq = \lambda\, dl$, and the current is $I = \lambda v$ (in agreement with Eq. (6.13) since $\lambda = nq$). Equation (6.1) tells us that the magnetic force on the piece is

$$d\mathbf{F} = dq\,\mathbf{v}\times\mathbf{B} = (\lambda\, dl)(v\hat{\mathbf{v}})\times\mathbf{B} = (\lambda v)(dl\,\hat{\mathbf{v}})\times\mathbf{B}$$

$$\Longrightarrow \quad \boxed{d\mathbf{F} = I\,d\mathbf{l}\times\mathbf{B}} \tag{6.17}$$

The vector $d\mathbf{l}$ gives both the magnitude and direction of the small piece. The $F = I_2 B_1 l$ result in Eq. (6.14) is a special case of this result.

Example (Copper wire) Let's apply Eqs. (6.13) and (6.15) to the pair of wires in Fig. 6.4(a). They are copper wires 1 mm in diameter and 5 cm apart. In copper the number of conduction electrons per cubic meter, already mentioned in Chapter 4, is $8.45\cdot 10^{28}$, so the number of electrons per unit length of wire is $n = (\pi/4)(10^{-3}\text{ m})^2(8.45\cdot 10^{28}\text{ m}^{-3}) = 6.6\cdot 10^{22}\text{ m}^{-1}$. Suppose their mean drift velocity $\bar{v}$ is $0.3\text{ cm/s} = 0.003\text{ m/s}$. (Of course their random speeds are vastly greater.) The current in each wire is then

$$I = nq\bar{v} = (6.6\cdot 10^{22}\text{ m}^{-1})(1.6\cdot 10^{-19}\text{ C})(0.003\text{ m/s}) \approx 32\text{ C/s}.$$

The attractive force on a 20 cm length of wire is

$$F = \frac{\mu_0 I^2 l}{2\pi r} = \frac{(4\pi\cdot 10^{-7}\text{ kg m/C}^2)(32\text{ C/s})^2(0.2\text{ m})}{2\pi(0.05\text{ m})} \approx 8\cdot 10^{-4}\text{ N}. \tag{6.18}$$

This result of $8\cdot 10^{-4}$ N is not an enormous force, but it is easily measurable. Figure 6.4(b) shows how the force on a given length of conductor could be observed.

Recall that the μ_0 in Eq. (6.18) can alternatively be written as $1/\epsilon_0 c^2$. The c^2 in the denominator reminds us that, as we discovered in Chapter 5, the magnetic force is a relativistic effect, strictly proportional to v^2/c^2 and traceable to a Lorentz contraction. And with the v in the above example less than the speed of a healthy ant, it is causing a quite respectable force! The explanation is the immense amount of negative charge the conduction electrons represent, charge that ordinarily is so precisely neutralized by positive charge that we hardly notice it. To appreciate that,

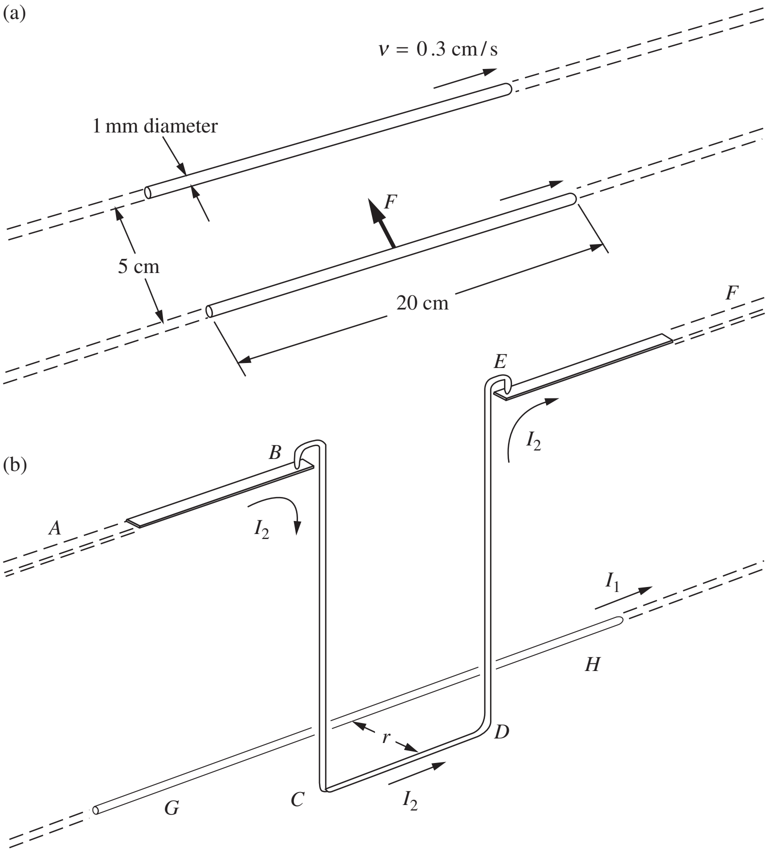


Figure 6.4.
(a) The current in each copper wire is 32 amps, and the force F on the 20 cm length of conductor is $8 \cdot 10^{-4}$ newtons. (b) One way to measure the force on a length of conductor. The section $BCDE$ swings like a pendulum below the conducting pivots. The force on the length CD due to the field of the straight conductor GH is the only force deflecting the pendulum from the vertical.

consider the force with which our wires in Fig. 6.4 would repel one another if the charge of the $6.6 \cdot 10^{22}$ electrons per meter were *not* neutralized at all. As an exercise you can show that the force is just c^2/v^2 times the force we calculated above, or roughly 4×10^{15} tons per meter of wire. So full of electricity is all matter! If all the electrons in just one raindrop were removed from the earth, the whole earth's potential would rise by several million volts.

Matter in bulk, from raindrops to planets, is almost exactly neutral. You will find that any piece of it much larger than a molecule contains nearly the same number of electrons as protons. If it didn't, the resulting electric field would be so strong that the excess charge would be irresistibly blown away. That would happen to electrons in our copper wire even if the excess of negative charge were no more than 10^{-10} of the total. A magnetic field, on the other hand, cannot destroy itself in this way. No matter how strong it may be, it exerts no force on a stationary charge. That is why forces that arise from the *motion* of electric charges can dominate the scene. The second term on the right in Eq. (6.1) can be much larger than the first. Thanks to that second term, an electric motor can start your car. In the atomic domain, however, where the coulomb force between pairs of charged particles comes into play, magnetic forces do take second place relative to electrical forces. They are weaker, generally speaking, by just the factor we should expect, the square of the ratio of the particle speed to the speed of light.

Inside atoms we find magnetic fields as large as 10 tesla (or 10^5 gauss). The strongest large-scale fields easily produced in the laboratory are on that order of magnitude too, although fields up to several hundred tesla have been created for short times. In ordinary electrical machinery, electric motors for instance, 1 tesla (or 10^4 gauss) would be more typical.[4] Magnetic resonance imaging (MRI) machines also operate on the order of 1 tesla. A magnet on your refrigerator might have a field of around 10 gauss. The strength of the earth's magnetic field is a few tenths of a gauss at the earth's surface, and presumably many times stronger down in the earth's metallic core where the currents that cause the field are flowing. We see a spectacular display of magnetic fields on and around the sun. A *sunspot* is an eruption of magnetic field with local intensity of a few thousand gauss. Some other stars have stronger magnetic fields. Strongest of all is the magnetic field at the surface of a neutron star, or pulsar, where in some cases the intensity is believed to reach the hardly conceivable range of 10^{10} tesla. On a vaster scale, our galaxy is pervaded by magnetic fields that extend over thousands of light years of interstellar space. The field strength can be deduced from observations in radioastronomy. It is a few microgauss – enough to make the magnetic field a significant factor in the dynamics of the interstellar medium.

6.2 Some properties of the magnetic field

The magnetic field, like the electric field, is a device for describing how charged particles interact with one another. If we say that the magnetic field at the point (4.5, 3.2, 6.0) at 12:00 noon points horizontally in the negative y direction and has a magnitude of 5 gauss, we are making a statement about the acceleration a moving charged particle at that point in space-time would exhibit. The remarkable thing is that a statement of this form, giving simply a vector quantity $\mathbf{B}$, says all there is to say. With it one can predict uniquely the velocity-dependent part of the force on *any* charged particle moving with *any* velocity. It makes unnecessary any further description of the other charged particles that are the sources of

[4] Nikola Tesla (1856–1943), the inventor and electrical engineer for whom the SI unit was named, invented the alternating-current induction motor and other useful electromagnetic devices. Gauss's work in magnetism was concerned mainly with the earth's magnetic field. Perhaps this will help you to remember which is the larger unit. For small magnetic fields, it is generally more convenient to work with gauss than with tesla, even though the gauss technically isn't part of the SI system of units. This shouldn't cause any confusion; you can quickly convert to tesla by dividing by (exactly) 10^4. If you're wary about leaving the familiar ground of SI units, feel free to think of a gauss as a deci-milli-tesla.

the field. In other words, if two quite different systems of moving charges happen to produce the same **E** and **B** at a particular point, the behavior of any test particle at the point would be exactly the same in the two systems. It is for this reason that the concept of field, as an intermediary in the interaction of particles, is useful. And it is for this reason that we think of the field as an independent entity.

Is the field more, or less, real than the particles whose interaction, as seen from our present point of view, it was invented to describe? That is a deep question which we would do well to set aside for the time being. People to whom the electric and magnetic fields were vividly real – Faraday and Maxwell, to name two – were led thereby to new insights and great discoveries. Let's view the magnetic field as concretely as they did and learn some of its properties.

So far we have studied only the magnetic field of a straight wire or filament of steady current. The field direction, we found, is everywhere perpendicular to the plane containing the filament and the point where the field is observed. The magnitude of the field is proportional to $1/r$. The field lines are circles surrounding the filament, as shown in Fig. 6.5. The sense of direction of **B** is determined by our previously adopted convention about the vector cross-product, by the (arbitrary) decision to write the second term in Eq. (6.1) as $q\mathbf{v} \times \mathbf{B}$, and by the *physical fact* that a positive charge moving in the direction of a positive current is attracted to it rather than repelled. These are all consistent if we relate the direction of **B** to the direction of the current that is its source in the manner shown in Fig. 6.5. Looking in the direction of positive current, we see the **B** lines curling clockwise. Or you may prefer to remember it as a right-hand-thread relation. Point your right thumb in the direction of the current and your fingers will curl in the direction of **B**.

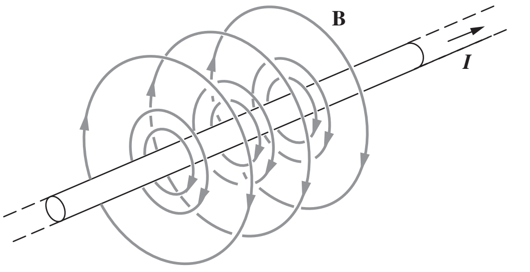
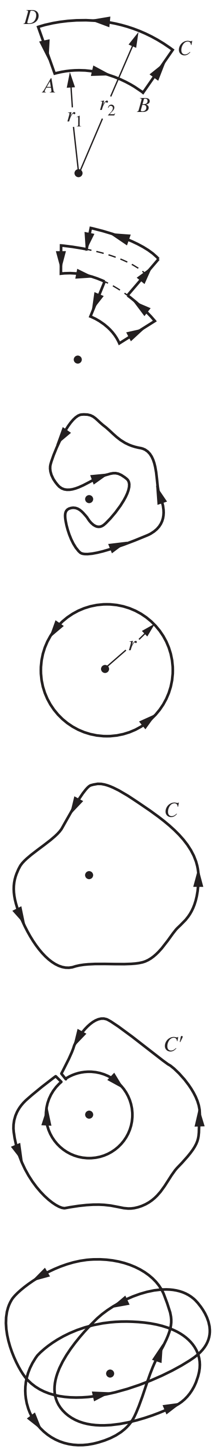


Figure 6.5.
Magnetic field lines around a straight wire carrying current.

Let's look at the line integral of **B** around a closed path in this field. (Remember that a similar inquiry in the case of the electric field of a point charge led us to a simple and fundamental property of all electrostatic fields, that $\int \mathbf{E} \cdot d\mathbf{s} = 0$ around a closed path, or equivalently that $\mathrm{curl}\,\mathbf{E} = 0$.) Consider first the path *ABCD* in Fig. 6.6(a). This lies in a plane perpendicular to the wire; in fact, we need only work in this plane, for **B** has no component parallel to the wire. The line integral of **B** around the path shown is zero, for the following reason. Paths *BC* and *DA* are perpendicular to **B** and contribute nothing. Along *AB*, **B** is stronger in the ratio r_2/r_1 than it is along *CD*; but *CD* is longer than *AB* by the same factor, for these two arcs subtend the same angle at the wire. So the two arcs give equal and opposite contributions, and the whole integral is zero.

It follows that the line integral is also zero on any path that can be constructed out of radial segments and arcs, such as the path in Fig. 6.6(b). From this it is a short step to conclude that the line integral is zero around *any* path that does not enclose the wire. To smooth out the corners we would only need to show that the integral around a



(a) Path lying in plane perpendicular to wire

(b) Path constructed of radial segments and arcs

(c) Path that does not enclose the wire

(d) Circular path enclosing wire

(e) Crooked path enclosing wire

(f) Circular and crooked path *not* enclosing wire

(g) Loop of N turns enclosing wire

Figure 6.6.
The line integral of the magnetic field $\mathbf{B}$ over any closed path depends only on the current enclosed.

little triangular path vanishes. The same step was involved in the case of the electric field.

A path that does not enclose the wire is one like the path in Fig. 6.6(c), which, if it were made of string, could be pulled free. The line integral around any such path is zero.

Now consider a circular path that encloses the wire, as in Fig. 6.6(d). Here the circumference is $2\pi r$, and the field is $\mu_0 I/2\pi r$ and everywhere parallel to the path, so the value of the line integral around this particular path is $(2\pi r)(\mu_0 I/2\pi r)$, or $\mu_0 I$. We now claim that *any* path looping once around the wire must give the same value. Consider, for instance, the crooked path C in Fig. 6.6(e). Let us construct the path C' in Fig. 6.6(f) made of a path like C and a circular path, but *not* enclosing the wire. The line integral around C' must be zero, and therefore the integral around C must be the negative of the integral around the circle, which we have already evaluated as $\mu_0 I$ in magnitude. The sign will depend in an obvious way on the sense of traversal of the path. Our general conclusion is:

$$\boxed{\int \mathbf{B} \cdot d\mathbf{s} = \mu_0 \times (\text{current enclosed by path})} \qquad \text{(Ampère's law)}. \tag{6.19}$$

This is known as *Ampère's law.* It is valid for *steady* currents. In the Gaussian analog of this expression, the μ_0 is replaced with $4\pi/c$, which quickly follows from a comparison of Eqs. (6.6) and (6.10).

Equation (6.19) holds when the path loops the current filament once. Obviously a path that loops it N times, like the one in Fig. 6.6(g), will just give N times as big a result for the line integral.

The magnetic field, as we have emphasized before, depends only on the rate of charge transport, the number of units of charge passing a given point in the circuit, per second. Figure 6.7 shows a circuit with a current of 5 milliamperes. The average velocity of the charge carriers ranges from 10^{-6} m/s in one part of the circuit to 0.8 times the speed of light in another. The line integral of $\mathbf{B}$ over a closed path has the same value around every part of this circuit, namely

$$\int \mathbf{B} \cdot d\mathbf{s} = \mu_0 I = \left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)\left(0.005\,\frac{\text{C}}{\text{s}}\right) = 6.3 \cdot 10^{-9}\,\frac{\text{kg m}}{\text{C s}}. \tag{6.20}$$

You can check that these units are the same as tesla-meter, which they must be, in view of the left-hand side of this equation.

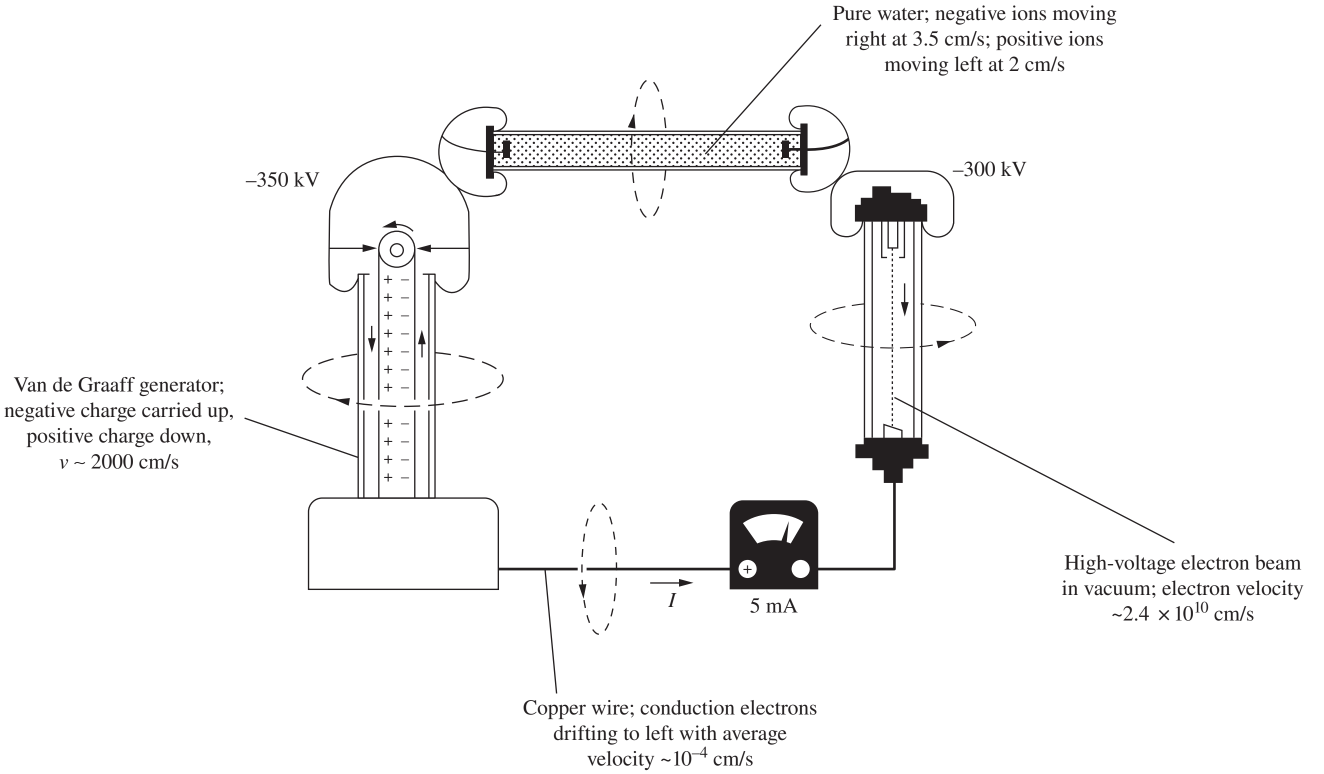


Figure 6.7.
The line integral of **B** has precisely the same value around every part of this circuit, although the velocity of the charge carriers is quite different in different parts.

What we have proved for the case of a long straight filament of current clearly holds, by superposition, for the field of any system of straight filaments. In Fig. 6.8 several wires are carrying currents in different directions. If Eq. (6.19) holds for the magnetic field of one of these wires, it must hold for the total field, which is the vector sum, at every point, of the fields of the individual wires. That is a pretty complicated field. Nevertheless, we can predict the value of the line integral of **B** around the closed path in Fig. 6.8 merely by noting which currents the path encircles, and in which sense.

Example (Magnetic field due to a thick wire) We know that the magnetic field outside an infinitesimally thin wire points in the tangential direction and has magnitude $B = \mu_0 I/2\pi r$. But what about a thick wire? Let the wire have radius R and carry current I with uniform current density; the wire may be viewed as the superposition of a large number of thin wires running parallel to each other. Find the field both outside and inside the wire.

Solution Consider an Amperian loop (in the spirit of a Gaussian surface) that takes the form of a circle with radius r around the wire. Due to the cylindrical symmetry, **B** has the same magnitude at all points on this loop. Also, **B** is

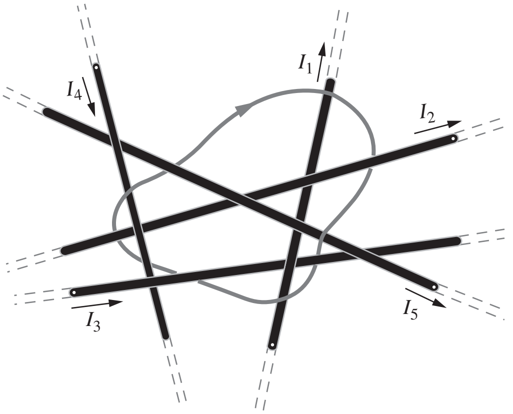


Figure 6.8.
A superposition of straight current filaments. The line integral of $\mathbf{B}$ around the closed path, in the direction indicated by the arrowhead, is equal to $\mu_0(-I_4 + I_5)$.

tangential; it has no radial component, due to the symmetric nature of the thin wires being superposed. So the line integral $\int \mathbf{B}\cdot d\mathbf{s}$ equals $B(2\pi r)$. Ampère's law then quickly gives $B = \mu_0 I/2\pi r$. We see that, outside a thick wire, the wire can be treated like a thin wire lying along the axis, as far as the magnetic field is concerned. This is the same result that holds for the electric field of a charged wire.

Now consider a point inside the wire. Since area is proportional to r^2, the current contained within a radius r inside the wire is $I_r = I(r^2/R^2)$. Ampère's law then gives the magnitude of the (tangential) field at radius r as

$$2\pi r B = \mu_0 I_r \implies B = \frac{\mu_0(Ir^2/R^2)}{2\pi r} = \frac{\mu_0 I r}{2\pi R^2} \qquad (r < R). \tag{6.21}$$

We have been dealing with long straight wires. However, we want to understand the magnetic field of any sort of current distribution – for example, that of a current flowing in a closed loop, a circular ring of current, to take the simplest case. Perhaps we can derive this field too from the fields of the individual moving charge carriers, properly transformed. A ring of current could be a set of electrons moving at constant speed around a circular path. But here that strategy fails us. The trouble is that an electron moving on a circular path is an *accelerated* charge, whereas the magnetic fields we have rigorously derived are those of charges moving with *constant velocity*. We shall therefore abandon our program of derivation at this point and state the remarkably simple fact: these more general fields *obey exactly the same law,* Eq. (6.19). The line integral of $\mathbf{B}$ around a bent wire is equal to that around a long straight wire carrying the same current. As this goes beyond anything we have so far deduced, we must look on it here as a postulate confirmed by the experimental tests of its implications.

You may find it unsettling that the validity of Ampère's law applied to an arbitrarily shaped wire simply has to be accepted, given that we have derived everything up to this point. However, this distinction between acceptance and derivation is illusory. As we will see in Chapter 9, Ampère's law is a special case of one of Maxwell's equations. Therefore, accepting Ampère's law is equivalent to accepting one of Maxwell's equations. And considering that Maxwell's equations govern all of electromagnetism (being consistent with countless experimental tests), accepting them is certainly a reasonable thing to do. Likewise, all of our derivations thus far in this book (in particular, the ones in Chapter 5) can be traced back to Coulomb's law, which is equivalent to Gauss's law, which in turn is equivalent to *another* one of Maxwell's equations. Therefore, accepting Coulomb's law is equivalent to accepting this other Maxwell equation. In short, everything boils down to Maxwell's equations sooner or later. Coulomb's law is no more fundamental than Ampère's law. We accepted the former long ago, so we shouldn't be unsettled about accepting the latter now.

To state Ampère's law in the most general way, we must talk about volume distributions of current. A general steady current distribution is described by a current density $\mathbf{J}(x, y, z)$ that varies from place to place but is constant in time. A current in a wire is merely a special case in which $\mathbf{J}$ has a large value within the wire but is zero elsewhere. We discussed volume distributions of current in Chapter 4, where we noted that, for time-independent currents, $\mathbf{J}$ has to satisfy the continuity equation, or conservation-of-charge condition,

$$\operatorname{div} \mathbf{J} = 0. \tag{6.22}$$

Take any closed curve C in a region where currents are flowing. The total current enclosed by C is the flux of $\mathbf{J}$ through the surface spanning C, that is, the surface integral $\int_S \mathbf{J} \cdot d\mathbf{a}$ over this surface S (see Fig. 6.9). A general statement of the relation in Eq. (6.19) is therefore

$$\int_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{a}. \tag{6.23}$$

Let us compare this with Stokes' theorem, which we developed in Chapter 2:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{a}. \tag{6.24}$$

We see that a statement equivalent to Eq. (6.23) is this:

$$\boxed{\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J}} \tag{6.25}$$

This is the differential form of Ampère's law, and it is the simplest and most general statement of the relation between the magnetic field and the moving charges that are its source. As with Eq. (6.19), the Gaussian analog of this expression has the μ_0 replaced by $4\pi/c$. Note that the form of $\mathbf{J}$ in Eq. (6.25) guarantees that Eq. (6.22) is satisfied, because the divergence of the curl is always zero (see Exercise 2.78).

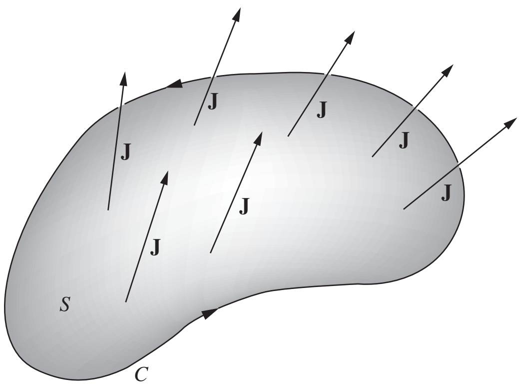


Figure 6.9.
$\mathbf{J}$ is the local current density. The surface integral of $\mathbf{J}$ over S is the current enclosed by the curve C.

Example (Curl of B for a thick wire) For the above "thick wire" example, verify that $\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J}$ both inside and outside the wire.

Solution We can use the expression for the curl in cylindrical coordinates given in Eq. (F.2) in Appendix F. The only nonzero derivative in the expression is $\partial(rA_\theta)/\partial r$, so outside the wire we have

$$\operatorname{curl} \mathbf{B} = \hat{\mathbf{z}} \frac{1}{r} \frac{\partial(rB_\theta)}{\partial r} = \hat{\mathbf{z}} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\mu_0 I}{2\pi r} \right) = 0, \tag{6.26}$$

which is correct because there is zero current density outside the wire. For the present purposes, the only relevant fact about the external field is that it is proportional to $1/r$.

Inside the wire we have

$$\operatorname{curl} \mathbf{B} = \hat{\mathbf{z}} \frac{1}{r} \frac{\partial(rB_\theta)}{\partial r} = \hat{\mathbf{z}} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\mu_0 I r}{2\pi R^2} \right) = \hat{\mathbf{z}} \mu_0 \frac{I}{\pi R^2} = \mu_0(\hat{\mathbf{z}} J) = \mu_0 \mathbf{J}, \tag{6.27}$$

as desired.

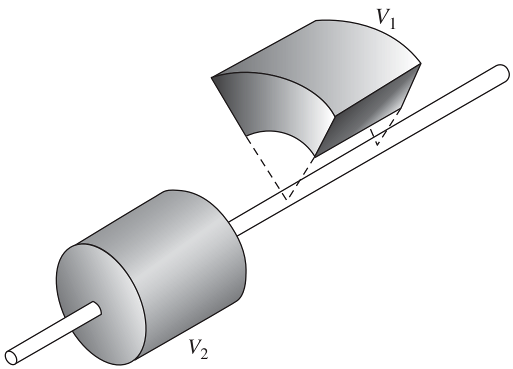


Figure 6.10.
There is zero net flux of **B** out of either box.

Equation (6.25) by itself is not enough to determine $\mathbf{B}(x, y, z)$, given $\mathbf{J}(x, y, z)$, because many different vector fields could have the same curl. We need to complete it with another condition. We had better think about the divergence of **B**. Going back to the magnetic field of a single straight wire, we observe that the divergence of that field is zero. You can't draw a little box anywhere, even one enclosing the wire, that will have a net outward or inward flux. It is enough to note that the boxes V_1 and V_2 in Fig. 6.10 have no net flux and can shrink to zero without developing any. (The $1/r$ dependence of B isn't important here. All that matters is that **B** points in the tangential direction and that its magnitude is independent of θ.) For this field then, $\text{div}\,\mathbf{B} = 0$, and hence also for all superpositions of such fields. Again we postulate that the principle can be extended to the field of any distribution of currents, so that a companion to Eq. (6.22) is the condition

$$\boxed{\text{div}\,\mathbf{B} = 0} \tag{6.28}$$

You can quickly check that this relation holds for the wire in the above example, both inside and outside, by using the cylindrical-coordinate expression for the divergence given in Eq. (F.2) in Appendix F; the only nonzero component of **B** is B_θ, but $\partial B_\theta/\partial\theta = 0$.

We are concerned with fields whose sources lie within some finite region. We won't consider sources that are infinitely remote and infinitely strong. Under these conditions, **B** goes to zero at infinity. With this proviso, we have the following theorem.

Theorem 6.1 *Assuming that* **B** *vanishes at infinity, Eqs. (6.25) and (6.28) together determine* **B** *uniquely if* **J** *is given.*

Proof Suppose both equations are satisfied by two different fields $\mathbf{B}_1$ and $\mathbf{B}_2$. Then their difference, the vector field $\mathbf{D} = \mathbf{B}_1 - \mathbf{B}_2$, is a field with zero divergence and zero curl everywhere. What could it be like? Having zero curl, it must be the gradient[5] of some potential function $f(x, y, z)$, that is, $\mathbf{D} = \nabla f$. But $\nabla \cdot \mathbf{D} = 0$, too, so $\nabla \cdot \nabla f$ or $\nabla^2 f = 0$ everywhere. Over a sufficiently remote enclosing boundary, f must take on some constant value f_0, because $\mathbf{B}_1$ and $\mathbf{B}_2$ (and hence **D**) are essentially zero very far away from the sources. Since f satisfies Laplace's equation everywhere inside that boundary, it cannot have a maximum or a minimum anywhere in that region (see Section 2.12), and so it must have the value f_0 everywhere. Hence $\mathbf{D} = \nabla f = 0$, and $\mathbf{B}_1 = \mathbf{B}_2$. □

The fact that a vector field is uniquely determined by its curl and divergence (assuming that it goes to zero at infinity) is known as the

[5] This follows from our work in Chapter 2. If $\text{curl}\,\mathbf{D} = 0$, then the line integral of **D** around any closed path is zero. This implies that we can uniquely define a potential function f as the line integral of **D** from an arbitrary reference point. It then follows that **D** is the gradient of f.

Helmholtz theorem. We proved this theorem in the special case where the divergence is zero.

In the case of the electrostatic field, the counterparts of Eqs. (6.25) and (6.28) were

$$\boxed{\text{curl}\,\mathbf{E} = 0} \qquad \text{and} \qquad \boxed{\text{div}\,\mathbf{E} = \frac{\rho}{\epsilon_0}} \tag{6.29}$$

In the case of the electric field, however, we could begin with Coulomb's law, which expressed directly the contribution of each charge to the electric field at any point. Here we shall have to work our way back to some relation of that type.[6] We shall do so by means of a *potential function*.